Application of Delay Differential Equations in the Model of the Relationship Between Unemployment and Inflation

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Abstract:

Purpose of the article: The article deals with a formulation of the dynamic model which expresses the relationship between the rate of inflation and the rate of unemployment more accurately and which has become to be known as the Phillips curve.

Methodology/methods: The economic theory in question, which is an essential basis of all economic models (Phillips curve, hysteresis of unemployment,...) is sufficiently explained in the introductory chapters and serves as the basis for formulating relationships of the economic quantities which are being explored. Here, there are applied methods of analysis, synthesis, dynamic modelling and differential equations.

Scientific aim: The classic model is limited to relations and mutual connections in time t only and eliminates the influence of factors from periods preceding time t. Mathematically, the model leads to a system of common differential equations. The aim of submitted the article is elimination of the simplifications by constructing a more realistic model, which takes into consideration the data from previous periods, and create a new model expressed by a system of two delay differential equations.

Findings: The new model can be solved (under various circumstances) and constructed even on the above mentioned more complex conditions and the impact of particular parameters of the model on its solution can be observed.

Conclusions: The original classic model of the relationship between the rate of inflation and the rate of unemployment which led to the Phillips curve has been replaced in this article by a new model expressing the real economic situation more precisely while respecting the influence of the history of the factors taken into account using so-called delay differential equations.

Keywords: differential equations, unemployment, inflation, Phillips curve, Hysteresis of unemployment

JEL Classification: C02, C69, E24, E27

Introduction

Unemployment, its causes, consequences and ways of dealing with it have been more or less intensively debated for more than 150 years both among specialists and among laymen. The main goal of most governments and politicians in economics is to reach a low rate of inflation as well as a low rate of unemployment. Historical experience, however, shows that such a goal is not achievable in the long run.

The classic economic model of the relationship between the rate of unemployment u and the rate of inflation π is denoted by a function whose graphic representation is so-called Phillips curve. It has become one of the tools which are still used to forecast the rate of inflation in the future. The model is limited to relations and mutual connections in time t only and eliminates the influence of factors from periods preceding time t. Mathematically, the model leads to a system of common differential equations.

The reasoning submitted in the article eliminates the simplifications by constructing a more realistic model, which takes into consideration the data from previous periods, and creates a new model expressed by a system of two delay differential equations.

1. Phillips curve

One of the tools which is still used to forecast the rate of inflation in the future is so-called Phillips curve. Its author is A.W. Phillips, a New Zealand born economist, wrote a paper in 1958 in which he describes how he observed an inverse relationship between hourly wage changes and unemployment in the British economy. In his article *The Relation between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861–1957* (Phillips, 1958), he analysed the relationship between the rate of unemployment and the hourly wage rate changes.

The curve describes a mutual inverse relationship between the rate of unemployment u and the rate of nominal hourly wage growth w. In other words, Phillips curve says that when the rate of unemployment is high, nominal wages grow more slowly, while if the rate of unemployment is low, nominal wages grow faster. The point where the Phillips curve intersects the x-axis is the so-called natural rate of unemployment.

Similar patterns were found in other countries and in 1960 Paul Samuelson a Robert Sollow "took" Phillips work, replaced the rate of wage changes with inflation and made the explicit link assuming that when inflation was high, unemployment was low, and vice-versa.. Samuelson a Solow (Samuelson and Solow, 1960) showed the empirical validity of the relationship on US data. Moreover, they argued that there is a positive correlation between nominal hourly wage changes and inflation, which made it possible to interpret the original Phillips relationship as a relationship between price inflation and the rate of unemployment.

The inverse relationship between the rate of inflation and the rate of unemployment is graphed by the modified Phillips curve. This curve also intersects the x-axis, the axis of the unemployment rate, at the point corresponding with the so-called natural unemployment rate $\bar{u}$. In this case, the natural unemployment rate is achieved at the zero rate of inflation, i.e. at a stable price level.

Milton Friedman later modified the Phillips curve. Friedman did not deny the interchange ability of inflation and unemployment in the short run. However, he regarded such interchange ability as a consequence of so-called money illusion. Therefore Friedman claims that Phillips curve is vertical in the long run. In the long term, unemployment remains at the level of so-.called natural unemployment rate which corresponds with the economic output at the level of potential output. The ensuing economic development in the 1970s corresponded to Milton Friedman's theory of the natural unemployment rate (Friedman, 1968). The natural unemployment rate is often quoted in the literature as an indicator of NAIRU (non-accelerating inflation rate of unemployment) which indicates a balanced rate of unemployment still consistent with stable inflation.

In paper (Cross, 1987), R. O. Cross deals with the natural unemployment rate. In this paper, two models are used: in one the rate of inflation helps determine the natural rate of unemployment, as proposed by Friedman; in the other the process of moving from one inflation rate to another helps shape the natural rate, according to the hysteresis effect proposed by Phelps.

The natural unemployment rate hypothesis implies a fact that if inflation has become stable and does not decrease any longer, it should oscillate around its natural rate, which for example Gordon (Gordon, 1989) identifies with the non-accelerating inflation rate of unemployment, i.e. NAIRU.

What should not be ignored, though, is the fact that mainly the European NAIRU increased considerably during the 1980s. Some economists claim, e.g. O. Blanchard (Blanchard, 2002), that a long-term high rate of unemployment tends to increase the natural unemployment rate. The paper (McNelis, 2003) applies neural network methodology to inflation forecasting in the Euro-area and the USA. McNelis observes in the paper that neural network methodology outperforms linear forecasting methods for the Euro Area at forecast horizons of one, three, and six month horizons, while the linear model is preferable for US data.

Although the Phillips curve remains to be seen as a controversial topic among economists, most of them now admit the concept of a short-term relationship between inflation and unemployment. The relationship is commonly explained to arise from slow adaptation of prices in a short term. Politicians may benefit from such a situation by means of many tools being at their disposal and they are able to influence the combination of inflation and unemployment in the short run.

2. Hysteresis of unemployment in modern literature

Gordon (Gordon, 1989) summarizes his explanation of the above mentioned phenomenon in two approaches – structural and hysteretic ones.

The structural approach saw and still sees the reason for growing NAIRU in specific obstacles on the supply side, while the hysteresis approach assumes that NAIRU follows the changes in current unemployment.

Economists use the term “hysteresis” to denote the persisting influence of past economic events. The reason is that temporary failures of a system may result in a permanent change in its description. The current value of an endogenous variable then may depend on past rather than current values of some exogenous variables.

The hysteretic model of unemployment is based on the Phillips curve with adaptive expectations. The main consequence of hysteresis in unemployment is that any rate of unemployment is consistent with stable inflation whose rate only depends on the past development of inflation and unemployment. The difference between the hysteresis hypothesis and the natural unemployment rate hypothesis can be illustrated on the relationship between unemployment and the course of a business cycle. The natural unemployment rate expects the return to a balanced state in the long run. By contrast, the existence of hysteresis assumes that unemployment is continuously influenced by cyclical ups and downs.

In (Pérez-Alonzo, 2011) Pérez-Alonzo proposes a new test for hysteresis based on a nonlinear unobserved components model. Observed unemployment rates are decomposed into a natural rate component and a cyclical component. A Monte Carlo simulation study shows the good performance of bootstrap algorithms. The bootstrap testing procedure is applied to data from Italy, France and the United States. Pérez-Alonzo finds evidence of hysteresis for all countries under study.

The existence of hysteresis and so-called Okun’s law were also tested in paper (Dinu, 2011), where, apart from other things, the authors tested the hypothesis of unemployment hysteresis in Romania during the period 1999–2008.

Fortin claims in (Fortin, 1991) that understanding and reducing hysteresis and developing a consensus-based income policy that has a chance of alleviating the dismal trade-off should be top research and policy priorities. The paper presents a review of the two competing paradigms explaining inflation and unemployment fluctuations and singles out the Keynesian Phillips curve as the clear empirical winner over its classical competitors. Re-estimation of the Canadian Phillips curve for 1957–1990 identifies a sharp structural shift towards a high degree of unemployment hysteresis after 1972.

The paper (Kula, 2010) examines the empirical validity of the hysteresis hypothesis in unemployment rates by education level in 17 OECD countries. These empirical findings provide evidence favourable to the non-stationarity of unemployment rates according to levels of primary and secondary education attainment in total unemployment, and therefore the existence of hysteresis for these levels of education. There is no evidence, however, of hysteresis for unemployment rates by tertiary education.

3. Hysteretic model of the relationship between unemployment and inflation

The hysteretic model of the relationship between unemployment and inflation contains a simple version of the natural unemployment rate hypothesis in time *t* (denoted $\bar{u}_t$), which connects inflation in time *t* (denoted πt) and unemployment rate in time *t* (denoted ut,). Further, the model considers the influence of unemployment rate and inflation rate in the period preceding time t by one time unit for simplicity, i.e. it also considers the influence of unemployment rate in time *t*–1, i.e. u_{t-1}, and the inflation rate in time t–1, i.e. π_{t-1}. This reasoning results in the following:

$$\pi_t = \alpha\pi_{t-1} + \beta\left(u_t - \overline{u}_t\right),$$

where parameter α denotes the inertia of the expected inflation development. This version of a simple Phillips curve (presented by Gordon (Gordon, 1989)) can be placed in the context of adaptive expectations. If α = 1, then NAIRU (balanced unemployment u_t) corresponds to a steady state in which $\pi_t = \pi_{t-1}$. Even values lower than one are consistent with the natural unemployment rate just because rationally acting agents may form their expectations with regard to a inflation rate drop. Allowing for the existence of hysteresis, we may define a rule which guides the development of a balanced unemployment rate $\bar{u}_t$ (represented by the degree of NAIRU):

$$\bar{u}_t = \eta u_{t-1} + z_t \ .$$

Hysteresis occurs when $\bar{u}_t$ depends on the delay value of the unemployment rate u_{t-1} and on microeconomic determinants represented by variable z_t. They can be identified with those listed by Friedmann in his natural unemployment rate hypothesis and we will assume that z_t is more or less invariable in time. By putting the relations together we get:

$$\pi_t = \alpha\pi_{t-1} + \beta\left(u_t - \eta u_{t-1} - z_t\right).$$

The ensuing transformation leads to the equation:

$$\pi_t = \alpha\pi_{t-1} + \beta(1-\eta)u_t + \beta\eta\left(u_t - u_{t-1}\right) - \beta z_t \ . \quad (1)$$

Note the theoretical aspects and implications brought about by the hypothesis of unemployment's hysteretic character. It is apparent that for $\eta = 1$ there is "full hysteresis". In such a case the unique $\bar{u}t$ will no longer exist and the balanced unemployment will be a completely variable quantity having no steady state value. Inflation in this case will not depend on the current unemployment rate but only on a change in unemployment. This, however, contradicts the natural unemployment rate hypothesis, which would tally with $\eta = 0$.

Unemployment inertia corresponds to the example where $0 < \eta < 1$, and it is in this case that NAIRU movement aims at a steady state, while the speed of this adjustment depends on parameter η (inertia rate).

The closer parameter η is to 1, the slower NAIRU's adjustment to the steady state will be and the lower the "inflation costs" will be of a demand-driven economic policy aiming at reducing the unemployment rate.

Parameters of such models are usually estimated by using classic estimation techniques of a Bayesian analysis or statistic filters. Apel in (Apel & Jansson, 1999) used a method combining statistic approaches with economic information – the multidimensional Hodrick-Prescott filter where residues of econometric equations capturing economic relationships related to a filtered variable are added in a special-purpose function which contains deviations of a real time series from a smoothed one, and changes in dynamics of smoothed data. Another option is to use the Kalman filter which, apart from direct incorporating of economic relationships, allows estimate or expert calibration of parameters. On the other hand, great flexibility in parameter setting and setting of statistic attributes of variables has drawbacks in the form of starting parameters and high optimization demands.

Another way to capture the dynamics of processes in a model is to describe a dynamical model by means of differential equations. Using this option we must understand time as a continuous variable. Formally, the dynamical system is described by a set of differential or difference equations.

Using principles of dynamical systems in economics has a long tradition and one of the definitions of an economic dynamic system comes from Paul Samuelson and R. Frisch (Gandalfo, 1971) says that: "A system is dynamical if behaviour over time is determined by functional equations in which variables at different points of time are involved in an essential way."

The term "dynamical economic models" is to be understood as such economic-mathematical models which include the behaviour of the analyzed system in time in their structures.

4. Dynamical model of the relationship between unemployment and inflation with a delay argument

If we introduce a new variable p which denotes the expected inflation rate and if we admit the hypothesis of adaptive expectations (adaptive formation of expected inflation unwinds from the changes in inflation in the past), then, instead of explaining the absolute value of π we can describe the changes in the course of time by means of the equation:

$$\frac{\mathrm{d}\pi}{\mathrm{d}t} = j\left(\pi_t - p\right),\ \left(0 < j < 1\right). \quad (2)$$

In the set of these two equations there are three variables, one of which, however, must be regarded as exogenous. Variables π and p shall be regarded as endogenous and variable u shall be regarded as exogenous. In order to explain the third variable u we shall add another equation in which the fact that inflation may impact unemployment will be taken into account.

If we consider the possibility of the impact of changes in monetary policy on the inflation rate, and if m shall denote the growth rate of money supply, then the change in the inflation rate can be defined:

$$\frac{d\pi}{dt} = -k\left(m - \pi_t\right),\ (k > 0) \tag{3}$$

In this relationship the change in the inflation rate is negatively dependant on the growth rate of money supply.

By substituting (1) in (2) and rearranging we obtain a system of two linear differential equations with delay argument where $x_1(t) = \pi_t (= \pi(t))$, $x_2(t) = u_t (= u(t))$ and $t \in [0, T]$:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= a_{11}x_1(t) + a_{12}x_2(t) + b_{11}x_1(t-1) + \\ &\quad + b_{12}x_2(t-1) + q_1 , \\ \frac{dx_2(t)}{dt} &= a_{21}x_1(t) + a_{22}x_2(t) + b_{21}x_1(t-1) + \\ &\quad + b_{22}x_2(t-1) + q_2 , \end{aligned} \tag{4}$$

where

$$\begin{aligned} &a_{11} = 0, a_{12} = -\beta j, a_{21} = k, a_{22} = 0, \\ &b_{11} = j\alpha, b_{12} = j\beta\eta, b_{21} = 0, b_{22} = 0, \\ &q_1 = -j(\beta z + p), q_2 = -km . \end{aligned}$$

The system can be briefly put down as

$$\frac{dx(t)}{dt} = Ax(t) + Bx(t) + q \tag{4'}$$

where

$$A = \begin{pmatrix} 0 & -j\beta \\ k & 0 \end{pmatrix}, B = \begin{pmatrix} j\alpha & j\beta\eta \\ 0 & 0 \end{pmatrix},$$

$$q = \begin{pmatrix} -j(\beta z + p) \\ -km \end{pmatrix},$$

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, t \in [0, T].$$

In order to explore economic relationships in the above model, we have to know conditions for the coefficient of system (4) which ensures the existence, or possibly unique existence of a solution to the system crossing the solution space in question at random point, *i.e.* meeting random so-called starting conditions.

$$x(t_0) = c_0 \tag{5}$$

where

$$t_0 \in [0, T], c_0 = \begin{pmatrix} c_{01} \\ c_{02} \end{pmatrix}, (c_{01}, c_{02}) \in R .$$

Next conditions for a non-negative solution, *i.e.*

$$x(t) \geq 0 \quad \text{for} \quad t \in [0, T] . \tag{6}$$

Conditions which guarantee the solution starting in values $x(0)$ to the same values in time T, *i.e.*

$$x(T) = x(0) , \tag{7}$$

(so-called periodic boundary condition).

5. Solution of a delay dynamic model

As equations (4) and (4') are equations with delay argument, they must be completed in both constituents by values in time preceding the values in time $t \in [0, T]$. In this case (the delay is given by a time unit $t = 1$) they are values of constituents of solution in the interval [–1, 0].

In accordance with the usual symbolism of differential equations with delay argument we shall denote them

$$\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}, t \in [-1, 0] .$$

The solution of system (4) or (4') meeting some of conditions 5–7 shall be completed by a condition of the behaviour of solution in the interval [–1,0], *i.e.*

$$x(t) = \varphi(t), t \in [-1, 0]. \tag{8}$$

The conditions (5–8) must be taken per constituents.

All these questions can be answered by a modern theory of functional differential equations, a special part of which covers a theory of linear differential equations with delay arguments.

Monograph (Kiguradze and Půža, 2003) contains a general theory allowing solutions to not only the above mentioned questions but also many others; the application of the theory to the above mentioned types of differential equations with delay argument, including the description of the construction of a required solution is dealt with in. (Kuchyňková and Maňásek, 2006) and all the sources quoted in it.

Conclusion

When modelling complex economic issues we often have to face the fact trade-offs between variables are chase in time. The dynamic character can be captured by including delay exogenous and endogenous variables in specifying the structure of a model.

Another way to include dynamic processes in models is to see time as a continuous variable and to describe dynamic models by means of differential equations.

The original classic model of the relationship between the rate of inflation and the rate of unemployment resulting in the Phillips curve has been replaced, in this paper, by a new model capturing more accurately the real economic situation and respecting the influence of the past of the factors considered.

The new model can be solved (under various circumstances) and constructed even on the above mentioned more complex conditions and the impact of particular parameters of the model on its solution can be observed.

High and unstable inflation has a negative impact on economy; therefore it must be fought against. In order to fight efficiently, we must fight in time, i.e. we must observe inflation and try to forecast it accurately, which the model can considerably contribute to.

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